Ordinal efficiency under the lens of duality theory

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Abstract

An allocation's ordinal efficiency deficit (OED) is defined as the greatest ordinal efficiency loss that can result from its application. More precisely, an allocation's OED is the negative of the greatest total amount by which it may be stochastically dominated by another feasible allocation. Thus, an allocation is ordinally efficient if and only if its OED is zero. Using this insight, we set up a linear program whose optimal objective value corresponds to a given allocation's OED. Furthermore, we show that the OED is a piecewise-linear convex function on the set of allocations. We use the optimal dual variables of the linear program to construct a profile of von Neumann-Morgenstern (vNM) utilities that is compatible with the underlying ordinal preferences, and which is a subgradient of the OED at the given allocation. When the given allocation is ordinally efficient, our analysis implies that it is ex-ante welfare maximizing at the constructed vNM profile, and we recover the ordinal efficiency theorem due to McLennan [10].

Keywords: random assignment, ordinal efficiency, linear programming, duality

JEL classifications: C61, D01, D60

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1 Introduction

In an influential paper, Bogomolnaia and Moulin [3] consider the probabilistic assignment of n objects to n agents. Agents are endowed with strict ordinal preferences over the set of objects and wish to be allocated the equivalent of one full object. To accommodate this fractional environment, Bogomolnaia and Moulin [3] adapt the familiar notion of Pareto efficiency to random assignments by introducing the concept of *ordinal efficiency.* A random assigment is ordinally efficient if agents cannot trade probability shares of objects to achieve a new random allocation that stochastically dominates the original one. Bogomolnaia and Moulin show that ordinal efficiency is equivalent to the acyclicity of a particular kind of binary relation between objects.[1] Abdulkadiroglu and Sonmez [1] provide a different characterization of ordinal efficiency based on a novel concept of dominated sets of assignments. In recent years, ordinal efficiency has been seen as an important benchmark in random assignment and has motivated the study and comparison of individual allocation mechanisms (Manea [7], Manea [9], Kesten [6], Che and Kojima [4]).

McLennan [10] offers a different characterization of ordinal efficiency. In particular, he considers the weak preference domain and shows that an allocation is ordinally efficient if and only if it is ex-ante welfare maximizing at some profile of von Neumann-Morgenstern (vNM) utilities, which are compatible with the underlying ordinal preferences. In his proof, he establishes and uses a new version of the separating hyperplane theorem. Manea [8] provides a simpler, constructive proof of McLennan's result that is based on the acyclicity of the binary relation discussed in [3] and [5]. The constructed profile of vNM utilities is related to a given weak representation of this (acyclic) binary relation.

Similarly to Manea [8], our work provides a simple constructive proof of McLennan's characterization. Given a feasible allocation and a profile of weak preferences, we define this allocation's ordinal efficiency deficit (with respect to the given preference profile) as the greatest total ordinal efficiency loss that can result from its application. More precisely, an allocation's OED is the negative of the greatest total amount by which it may be stochastically dominated by another feasible allocation. For example, consider an economy with three agents (1, 2 and 3), and three objects (a, b and c). Agent 1 strictly prefers object a to b and b to c; agent 2 strictly prefers b to c and c to a; and agent 3 c to a and a to b (preferences are complete, reflexive, and transitive). Consider the fractional allocation where all three agents are awarded a share of 1/3 of all houses. This allocation is clearly ordinally inefficient, and is strictly dominated by many feasible allocations. Indeed, the only ordinally efficient allocation in this economy is the deterministic outcome in which agent 1 gets all of a, 2 all of b, and 3 all of c. Focusing on agent 1 and his most preferred object a, we see that the ordinal efficiency loss here is equal to -2/3; examining the same agent and the set of his first and second-most preferred objects, a and b, the efficiency loss is -1/3. Adding the two together

[1] Katta and Sethuraman [5] extend Bogomolnaia and Moulin's analysis to the weak preference domain.

yields -1. Examining each agent in this way, we establish that the original allocation's OED with respect to the given preference profile is -3.[2]

Contribution. Clearly, an allocation is ordinally efficient if and only if its OED is zero. Using this insight, we set up a linear program (LP) whose optimal objective value corresponds to a given allocation's OED. Furthermore, we show that the OED is a piecewise-linear convex function on the set of allocations. We use the optimal dual variables of the previously mentioned LP to exhibit a profile of vNM utilities that is compatible with the underlying ordinal preferences, and which is a subgradient of the OED at the given allocation. This result is, in a sense, more general than the ordinal efficiency theorem and points to a deeper connection between ordinal and cardinal measures of efficiency in random-assignment problems. Indeed, when a given allocation is ordinally efficient our analysis implies that it is ex-ante welfare maximizing at the constructed vNM profile, and we recover McLennan's result. It is our hope that the simplicity of our LP-based approach may prove helpful in thinking about related problems in the growing field of random assignment.

Structure of the Paper. The structure of the paper is as follows. Section 2 introduces the model, and Section 3 provides a proof of McLennan's [10] ordinal efficiency welfare theorem that is based on LP duality. Section 4 generalizes the approach pursued in Section 3 to arbitrary (i.e., non ordinally efficient) allocations. It introduces the concept of an allocation's OED and discusses the interpretation of the constructed vNM utility profiles as subgradients of the OED at a given allocation. Section 5 provides concluding remarks.

2 Model Description

Consider an economy with a set N of n agents and M of m objects indexed by $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$, respectively. Suppose without loss of generality that $m \geq n$, allowing for the possibility of "dummy" objects that correspond to not being assigned anything at all. Each agent i's preferences over the set of objects are expressed by the complete, reflexive, and transitive relation $\succeq_i$, and $\succeq$ denotes the economy-wide profile of preferences $\{\succeq_i\}_{i=1}^{n}$. If objects j_1 and j_2 are such that $j_1 \succeq_i j_2$ and $j_2 \succeq_i j_1$ then agent i is indifferent between them, and this is denoted by $j_1 \sim_i j_2$. If $j_1 \succeq_i j_2$, but $j_2 \not\succeq_i j_1$, then agent i strictly prefers object j_1 to j_2, and this is denoted by $j_1 \succ_i j_2$.

In what follows, a prime symbol following a given (column) vector denotes the vector's transpose. An *individual allocation* for agent i is a non-negative column vector $p_i = (p_{i1}, p_{i2}, \ldots, p_{im})'$ such that $\sum_j p_{ij} = 1$. An *allocation* $\mathbf{p} = (p'_1, p'_2, ..., p'_n)'$ is a concatenation of a set of individual allocations p_i

[2] For simplicity, from now on we omit making explicit the dependence of these concepts on the underlying preference profile except where necessary.

for $i = 1, 2, ..., n$ that satisfies $\sum_i p_{ij} \leq 1$ for all $j \in M$.[3] Let P denote the set of all allocations.

For agent i, an individual allocation p_i *dominates* another q_i at $\succeq_i$, denoted $p_i \succeq_i q_i$, whenever

$$\sum_{a \succeq_i j} p_{ia} \geq \sum_{a \succeq_i j} q_{ia}, \text{ for all } j \in M.$$

If at least one of the above inequalities is strict, then p_i *strictly dominates* q_i, which is denoted by $p_i \succ_i q_i$. The dominance relation defined on an individual allocation extends to its economy-wide equivalent in a natural way: an allocation $\mathbf{p}$ *dominates* an allocation $\mathbf{q}$ at $\succeq$ if $p_i \succeq_i q_i$ for every agent i; $\mathbf{p}$ *strictly dominates* $\mathbf{q}$ at $\succeq$ if $\mathbf{p}$ dominates $\mathbf{q}$, and if $p_i \succ_i q_i$ for some agent i. Bogomolnaia and Moulin [3] introduce the following efficiency criterion: An allocation $\mathbf{p}$ is said to be *ordinally efficient* at $\succeq$ if there does not exist an allocation $\mathbf{q}$ that strictly dominates it at $\succeq$.

A profile of von Neumann-Morgenstern (vNM) utility functions $\mathbf{u} = \big(u_i : M \to \Re, \;\; i \in N\big)$ is *compatible* with a profile of ordinal preferences $\succeq$ if

$$u_i(j_1) \geq u_i(j_2) \Leftrightarrow j_1 \succeq_i j_2, \;\; \forall j_1, j_2 \in M, \;\; \forall i \in N.$$

Finally, an allocation $\mathbf{p}$ is *ex-ante welfare maximizing* at a profile of vNM utilities $\mathbf{u}$ if it maximizes the social welfare function

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} u_i(j),$$

over the set of feasible allocations.

3 A Duality Proof of the Ordinal Efficiency Welfare Theorem

For simplicity, assume that preferences are strict (Remark 1 clarifies how the argument extends to the general case). In what follows, we use LP duality to provide a constructive proof of McLennan's ordinal efficiency welfare theorem.

Theorem 1 *(McLennan [10]) An allocation is ordinally efficient at $\succeq$ if and only if it is ex-ante welfare-maximizing at some profile of vNM utilities compatible with $\succeq$.*

Proof. Consider an allocation $\hat{\mathbf{p}} \in P$ and denote by $j_i(k)$ agent i's k'th most preferred object. Where applicable, let $\mathbf{0}$ denote a zero vector of appropriate dimension. Consider the following

[3] Note how the elements p_{ij} of $\mathbf{p}$ are positioned in lexicographic order. For reasons that will become apparent in Section 3, we avoid the more common representation of an allocation as a sub-stochastic matrix.

linear program (LP) in standard form:

$$\begin{aligned}
\min_{\mathbf{p},\mathbf{r},\mathbf{s}} \quad & \sum_{i=1}^{n}\sum_{k=1}^{m} -r_{ik} \\
\text{subject to:} \quad & \sum_{j \succeq_i j_i(k)} p_{ij} - r_{ik} = \sum_{j \succeq_i j_i(k)} \hat{p}_{ij}, \ \ \forall k \in \{1,2,...,m\}, \ \ \forall i \in N \\
& \sum_{j=1}^{m} p_{ij} = 1, \ \ \forall i \in N \\
& \sum_{i=1}^{n} p_{ij} + s_j = 1, \ \ \forall j \in M \\
& \mathbf{p} \geq \mathbf{0}, \ \ \mathbf{r} \geq \mathbf{0}, \ \ \mathbf{s} \geq \mathbf{0}.
\end{aligned} \tag{1}$$

By definition, the solution $(\mathbf{p},\mathbf{r},\mathbf{s}) = (\hat{\mathbf{p}},\mathbf{0},\hat{\mathbf{s}})$ (where $\hat{s}_j = 1 - \sum_{i=1}^{n} \hat{p}_{ij}$ for all $j \in M$), is feasible and establishes an upper bound of 0 for the problem's optimal cost (i.e., objective value).

Using the definition of ordinal efficiency, it is easy to see that $\hat{\mathbf{p}}$ is ordinally efficient if and only if the optimal solution $(\mathbf{p}^*,\mathbf{r}^*,\mathbf{s}^*)$ of the primal problem (1) satisfies $(\mathbf{p}^*,\mathbf{r}^*,\mathbf{s}^*) = (\hat{\mathbf{p}},\mathbf{0},\hat{\mathbf{s}})$, thus yielding an optimal cost of 0.

Taking the dual of (1), and letting $\mathbf{1}$ denote a unit vector of dimension $n \cdot m$, we obtain[4]

$$\begin{aligned}
\max_{\mathbf{x},\mathbf{y},\mathbf{z}} \quad & \sum_{i=1}^{n}\sum_{k=1}^{m} x_{ik} \sum_{j \succeq_i j_{i(k)}} \hat{p}_{ij} + \sum_{i=1}^{n} y_i + \sum_{j=1}^{m} z_j \\
\text{subject to:} \quad & \sum_{j=k}^{m} x_{ij} + y_i + z_{j_i(k)} \leq 0, \ \ \forall k \in \{1,2,...,m\}, \ \ \forall i \in N \\
& \mathbf{x} \geq \mathbf{1} \\
& \mathbf{y} \ \text{ free variable}, \ \ \mathbf{z} \leq \mathbf{0}.
\end{aligned} \tag{2}$$

By strong duality (see Theorem 4.4 in [2]), the primal problem has an optimal cost of 0 (which, as mentioned before, is equivalent to $\hat{\mathbf{p}}$ being ordinally efficient) if and only if the optimal solution of the dual problem (2), $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$, satisfies

$$\sum_{i=1}^{n}\sum_{k=1}^{m} \hat{x}_{ik} \sum_{j \succeq_i j_i(k)} \hat{p}_{ij} + \sum_{i=1}^{n} \hat{y}_i + \sum_{j=1}^{m} \hat{z}_j = 0. \tag{3}$$

Now, let $\hat{\mathbf{u}}$ denote a profile of von-Neumann Morgenstern (vNM) utilities such that

$$\hat{u}_i(j_i(k)) = \sum_{j=k}^{m} \hat{x}_{ij}, \ \ k \in \{1,2,...,m\}, \ \ i \in N. \tag{4}$$

Recall that since $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ is feasible, we must have $\hat{x}_{ik} \geq 1$ for all i,k. In combination with Eq. (4), this immediately establishes that $\hat{\mathbf{u}}$ is compatible with the ordinal preferences $\succeq$. Rearranging terms, Eq. (3) can be rewritten in the following way

$$\sum_{i=1}^{n}\sum_{k=1}^{m} \hat{u}_i(j_i(k))\hat{p}_{ij_i(k)} = -\left(\sum_{i=1}^{n} \hat{y}_i + \sum_{j=1}^{m} \hat{z}_j\right) \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{m} \hat{u}_i(j)\hat{p}_{ij} = -\left(\sum_{i=1}^{n} \hat{y}_i + \sum_{j=1}^{m} \hat{z}_j\right) \tag{5}$$

[4]For details see Chapter 4.2 in Bertsimas and Tsitsiklis [2].

Again by dual feasibility we must have

$$\begin{aligned} & m-k+1 \leq \hat{u}_i(j_{i(k)}) \leq -\big(\hat{y}_i + \hat{z}_{j_i(k)}\big), \ \ \forall k \in \{1,2,...,m\}, \ i \in N \\ \Rightarrow \ \ & 1 \leq u_i(j) \leq -\big(\hat{y}_i + \hat{z}_j\big), \ \ \forall j \in M, \ i \in N. \end{aligned} \tag{6}$$

Now, consider an arbitrary $\mathbf{p} \in P$. We have

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} \hat{u}_i(j)p_{ij} \ \ & \overset{(6)}{\leq} \ \ \sum_{i=1}^{n}\sum_{k=1}^{m} -\big(\hat{y}_i + \hat{z}_j\big)p_{ij} \overset{(1)\ (2)}{\leq} -\Bigg(\sum_{i=1}^{n}\hat{y}_i + \sum_{j=1}^{m}\hat{z}_j\Bigg) \\ & \overset{(5)}{=} \ \ \sum_{i=1}^{n}\sum_{j=1}^{m} \hat{u}_i(j)\hat{p}_{ij}. \end{aligned} \tag{7}$$

Thus, $\hat{\mathbf{p}}$ is ex-ante welfare maximizing at the vNM utility profile $\hat{\mathbf{u}}$, which is compatible with the ordinal preferences $\succeq$.

We have proved that if $\hat{\mathbf{p}}$ is ordinally efficient, then it is ex-ante welfare maximizing for some vNM utility profile that is compatible with the agents' ordinal preferences. The other direction is easily established (see Lemma 1 in [3]) so that the equivalence of the two statements follows. ■

It is easy to show that the optimal solution of the primal LP (1) always produces an ordinally efficient solution, independently of whether $\hat{\mathbf{p}}$ is ordinally efficient.

Proposition 1 *Consider the optimal solutions* $(\mathbf{p}^*, \mathbf{r}^*, \mathbf{s}^*)$ *and* $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ *of the primal (1) and dual (2) LPs, respectively. Moreover, consider the vNM utility profile* $\hat{\mathbf{u}}$*, that is a function of the optimal* $\hat{\mathbf{x}}$ *variables, given by Eq. (4). The following two statements hold:*

(a) The allocation $\mathbf{p}^*$ *is ordinally efficient, and*

(b) For all $i \in N$ *and* $j \in M$ *we have*

$$p^*_{ij} > 0 \Leftrightarrow \hat{u}_i(j) = -\big(\hat{y}_i + \hat{z}_j\big).$$

Proof. (a) As established in the proof of Theorem 1, if $\mathbf{r}^* = \mathbf{0}$ then $\mathbf{p}^* = \hat{\mathbf{p}}$ is ordinally efficient. So we focus on the case $\mathbf{r}^* \neq \mathbf{0}$ and suppose that $\mathbf{p}^*$ is not ordinally efficient. Then there exists a feasible allocation $\tilde{\mathbf{p}}$ which strictly dominates $\mathbf{p}^*$. Consider the solution $(\tilde{\mathbf{p}}, \tilde{\mathbf{r}}, \tilde{\mathbf{s}})$, where $\tilde{s}_j = 1 - \sum_{i=1}^{m} \tilde{p}_{ij}$ for all $j \in M$, and

$$\tilde{r}_{ik} = r^*_{ik} + \sum_{j \succeq_i j_{i(k)}} (\tilde{p}_{ij} - p^*_{ij}) \geq r^*_{ik}, \ \ k \in \{1,2,..,m\}, \ \ \forall i \in N. \tag{8}$$

The solution $(\tilde{\mathbf{p}}, \tilde{\mathbf{r}}, \tilde{\mathbf{s}})$ is easily seen to be feasible for the primal problem (1) as

$$\sum_{j \succeq_i j_i(k)} \tilde{p}_{ij} - \tilde{r}_{ik} = \sum_{j \succeq_i j_i(k)} \tilde{p}_{ij} - \Bigg[r^*_{ik} + \sum_{j \succeq_i j_i(k)} (\tilde{p}_{ij} - p^*_{ij})\Bigg] = \sum_{j \succeq_i j_i(k)} p^*_{ij} - r^*_{ik} = \sum_{j \succeq_i j_i(k)} \hat{p}_{ij}, \ \ \forall i, k,$$

while all other constraints are trivially satisfied. Since $\tilde{\mathbf{p}}$ strictly dominates $\mathbf{p}^*$, at least one of the inequalities given by Eq. (8) is strictly satisfied. This implies that the feasible solution $(\tilde{\mathbf{p}}, \tilde{\mathbf{r}}, \tilde{\mathbf{s}})$ yields a strictly smaller cost than $(\mathbf{p}^*, \mathbf{r}^*, \mathbf{s}^*)$ contradicting the latter's optimality.

(b) This part follows trivially from the complementary slackness conditions (see Theorem 4.5 in [2])

$$p^*_{ij_i(k)}\left(\sum_{j=k}^{m} \hat{x}_{ij} + \hat{y}_i + \hat{z}_{j_i(k)}\right) = 0,\ \ \forall k \in \{1, 2, ..., m\},\ i \in N.$$

■

Remark 1. It is straightforward to modify our argument when indifferences are allowed. Suppose agent i is allowed to be indifferent between various objects, so that he has $m_i \in \{1, .., m\}$ indifference classes. Here, an object is said to belong in agent i's k'th indifference class, $I_i(k)$, if it is among his k'th-most preferred. Consequently, we only introduce variables r_{ik} where $k = 1, ..., m_i$, and adapt the primal problem's relevant constraints to[5]

$$\sum_{j \succeq_i I_i(k)} p_{ij} - r_{ik} = \sum_{j \succeq_i I_i(k)} \hat{p}_{ij},\ \ \forall k \in \{1, 2, ..., m_i\},\ \ \forall i \in N.$$

The corresponding dual constraints are modified to

$$\sum_{j=k}^{m_i} x_{ij} + y_i + z_j \leq 0,\ \ \forall j \in I_i(k),\ \ \ k \in \{1, 2, ..., m_i\},\ \ \forall i \in N,$$

while the vNM utility profile $\hat{\mathbf{u}}$ is defined so that

$$\hat{u}_i(j) = \sum_{j=k}^{m_i} \hat{x}_{ij},\ \ \forall j \in I_i(k),\ \forall k \in \{1, 2, ..., m_i\},\ \forall i \in N.$$

The logic of the proof then carries over. Notice how the vNM utility profile $\hat{\mathbf{u}}$ assigns identical utility to objects over which an agent is indifferent.

4 A Subtler Connection Between Ordinal Efficiency and vNM Utilities

In this section, we make a more general connection between ordinal efficiency and the profile of vNM utilities discussed in Section 3. Indeed, duality theory lends the constructed vNM utility profile $\hat{\mathbf{u}}$ of Theorem 1 a novel economic interpretation, regardless of whether the candidate allocation $\hat{\mathbf{p}}$ is ordinally efficient. Throughout, we fix a preference profile $\succeq$ and suppress the explicit dependence

[5] Abusing notation, we denote the set of objects that are at least as preferred to agent i as those in his k'th indifference class by $j \succeq_i I_i(k)$.

of our results on the economy's preferences. Once again, we assume strict preferences; it is clear how the results extend to the weak domain. We begin by defining the concept of a subgradient that is pervasive in convex optimization.

Definition 1 *Let $f : \mathcal{X} \to \Re$ denote a convex function defined on a convex set $\mathcal{X}$. Let $\hat{x} \in \mathcal{X}$. A vector $\mathbf{v} \in \mathcal{X}$ is a subgradient of f at $\hat{x}$ if*

$$f(\hat{x}) + \mathbf{v} \cdot (x - \hat{x}) \leq f(x), \;\; \forall x \in \mathcal{X}.$$

Returning to our model, let $\mathbf{p} \in P$ and reorder its elements so that

$$\mathbf{p}_{ik} = p_{ij_i(k)}, \;\; \forall k \in \{1, 2, ..., m\}, \;\; i \in N.$$

Now, we define the vector-valued function $\mathbf{g} : P \to \Re^{n \cdot m}$, where $\mathbf{g}(\mathbf{p}) = \left(\mathbf{g}(\mathbf{p})'_{\mathbf{1}}, \mathbf{g}(\mathbf{p})'_{\mathbf{2}}, ..., \mathbf{g}(\mathbf{p})'_{\mathbf{n}}\right)'$, such that

$$\mathbf{g}(\mathbf{p})_{ik} = \sum_{j \succeq_i j_i(k)} p_{ij}, \;\; \forall k \in \{1, 2, .., m\}, \;\; i \in N. \tag{9}$$

Given column vectors $\mathbf{x_i} \in \Re^m$ for all $i \in N$, let $\mathbf{x} = \left(\mathbf{x}'_{\mathbf{1}}, \mathbf{x}'_{\mathbf{2}}, ..., \mathbf{x}'_{\mathbf{n}}\right)' \in \Re^{n \cdot m}$. Next, we define $\mathbf{u} : \Re^{n \cdot m} \to \Re^{n \cdot m}$ to be a vector-valued function, where $\mathbf{u}(\mathbf{x}) = \left(\mathbf{u}(\mathbf{x})'_{\mathbf{1}}, \mathbf{u}(\mathbf{x})'_{\mathbf{2}}, ..., \mathbf{u}(\mathbf{x})'_{\mathbf{n}}\right)'$, such that

$$\mathbf{u}(\mathbf{x})_{ik} = \sum_{j=k}^{m} x_{ij}, \;\; \forall k \in \{1, 2, .., m\}, \;\; i \in N. \tag{10}$$

Echoing the proof of Theorem 1, we can rearrange terms and establish the following identity

$$\mathbf{x}'\mathbf{g}(\mathbf{p}) = \mathbf{u}(\mathbf{x})'\mathbf{p}, \;\; \forall \mathbf{p} \in P, \;\; \mathbf{x} \in \Re^{n \cdot m}. \tag{11}$$

We use the primal problem (1) to define an allocation's *ordinal efficiency deficit* (OED) as the negative of the greatest amount by which it can be stochastically dominated by another feasible allocation. Or, equivalently, as the greatest ordinal efficiency loss that its application can result in. Let $F(\hat{\mathbf{p}})$ denote the feasible region of the primal problem (1) for a given $\hat{\mathbf{p}} \in P$ so that

$$F(\hat{\mathbf{p}}) = \left\{ (\mathbf{p}, \mathbf{r}, \mathbf{s}) \geq \mathbf{0} \;\middle|\; \sum_{j \succeq_i j_i(k)} p_{ij} - r_{ik} = \sum_{j \succeq_i j_i(k)} \hat{p}_{ij}, \; \sum_{j=1}^{m} p_{ij} = 1, \;\; \sum_{i=1}^{n} p_{ij} + s_j = 1, \;\; \forall i, j, k \right\}. \tag{12}$$

The OED of an allocation $\hat{\mathbf{p}}$ is defined as the optimal cost of the primal problem (1) when the allocation in the right-hand-side of the constraints constraints is given by $\hat{\mathbf{p}}$. Formally, it is denoted by a function $D : P \;\to \Re_{-}$ such that

$$D(\hat{\mathbf{p}}) = \min_{(\mathbf{p}, \mathbf{r}, \mathbf{s}) \in F(\hat{\mathbf{p}})} \sum_{i=1}^{n} \sum_{k=1}^{m} -r_{ik}.$$

Proposition 2 *The ordinal efficiency deficit $D(\cdot)$ is a piecewise-linear convex function on the set P.*

Proof. The argument follows closely Section 5.2 in Bertsimas and Tsitsiklis [2]. Let $\hat{\mathbf{p}} \in P$ and consider the associated dual LP (2). By strong duality, the dual's optimal cost is finite and equal to $D(\hat{\mathbf{p}})$. To make the dual feasible region a polyhedron (see Definition 2.1 in [2]), we substitute the free variable $\mathbf{y}$ by the difference of two non-negative variables $\mathbf{y}^+$ and $\mathbf{y}^-$. Since any real number can be written as the difference of two non-negative real numbers, the two problems yield the same optimal cost.

The set of (updated) dual constraints $\{\mathbf{x} \geq 1, \mathbf{y}^+ \geq 0, \mathbf{y}^- \geq 0, \mathbf{z} \leq \mathbf{0}\}$, ensures that the new dual feasible region is a polyhedron that does not contain a line (see Definition 2.12 in [2]). Consequently, Theorem 2.6 of [2] implies that the dual feasible region (unaltered by changes in $\hat{\mathbf{p}}$) contains at least one extreme point. Let $\left(\mathbf{x^k}, (\mathbf{y}^+)^\mathbf{k}, (\mathbf{y}^-)^\mathbf{k}, \mathbf{z^k}\right)$ for $k = 1, 2, ..., N$ be the extreme points of the dual feasible region. By Theorem 2.8 of [2], the optimum of the dual must be attained at some extreme point. Hence, we may write:

$$\begin{aligned} D(\hat{\mathbf{p}}) &= \max_{k=1,..,N} \left\{ \left(\mathbf{x^k}, (\mathbf{y}^+)^\mathbf{k} - (\mathbf{y}^-)^\mathbf{k}, \mathbf{z^k}\right)'(\mathbf{g}(\hat{\mathbf{p}}), \mathbf{1}, \mathbf{1}) \right\} \\ &= \max_{k=1,..,N} \left\{ \mathbf{u}(\mathbf{x^k})'\hat{\mathbf{p}} + \left((\mathbf{y}^+)^\mathbf{k} - (\mathbf{y}^-)^\mathbf{k}, \mathbf{z^k}\right)'(\mathbf{1}, \mathbf{1}) \right\}. \end{aligned} \tag{13}$$

Since the maximum of a set of linear (and therefore convex) functions is itself convex, the result follows. ∎

We are now ready to generalize the insights obtained in the proof of Theorem 1.

Theorem 2 *Consider a profile of preferences $\succeq$ and an allocation $\hat{\mathbf{p}} \in P$. Let $j_i(k)$ denote agent i's k'th-most preferred object. Suppose the vector $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ is an optimal solution of LP (2) and consider the vNM utility profile $\hat{\mathbf{u}}$ where*

$$\hat{u}_i(j_i(k)) = \sum_{j=k}^{m} \hat{x}_{ij}, \ \ \forall k \in \{1, 2, ..., m\}, \ i \in N.$$

This profile is (a) compatible with the underlying ordinal preferences, and (b) a subgradient of the ordinal efficiency deficit D at $\hat{\mathbf{p}}$.

Proof. Part (a) follows immediately from dual feasibility. We turn to part (b). The simple argument follows the proof of Theorem 5.2 in Bertsimas and Tsitsiklis [2]. First, recall our earlier notation

$$\mathbf{u}(\hat{\mathbf{x}})_{ik} = \sum_{j=k}^{m} \hat{x}_{ij}, \ \ \forall k \in \{1, 2, .., m\}, \ \ i \in N.$$

Strong duality implies that

$$(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})'(\mathbf{g}(\hat{\mathbf{p}}), \mathbf{1}, \mathbf{1}) = D(\hat{\mathbf{p}}) \overset{(11)}{\Rightarrow} \mathbf{u}(\hat{\mathbf{x}})'\hat{\mathbf{p}} + \sum_{i=1}^{n} \hat{y}_i + \sum_{j=1}^{m} \hat{z}_j = D(\hat{\mathbf{p}}).$$

Consider now an arbitrary $\tilde{\mathbf{p}} \in P$. By weak duality (see Theorem 4.3 in [2]), we have

$$\mathbf{u}(\hat{\mathbf{x}})'\tilde{\mathbf{p}} + \sum_{i=1}^{n} \hat{y}_i + \sum_{j=1}^{m} \hat{z}_j \leq D(\tilde{\mathbf{p}}).$$

Hence, we may conclude that

$$\mathbf{u}(\hat{\mathbf{x}})'(\tilde{\mathbf{p}} - \hat{\mathbf{p}}) \leq D(\tilde{\mathbf{p}}) - D(\hat{\mathbf{p}}), \ \ \forall \, \tilde{\mathbf{p}} \in P.$$

■

Remarks. Theorem 2 leads to some interesting observations. Suppose we have an allocation $\hat{\mathbf{p}}$, its associated optimal dual variables $\hat{\mathbf{x}}$, and the resulting profile of vNM utilities $\hat{\mathbf{u}} = \mathbf{u}(\hat{\mathbf{x}})$. Consider an arbitrary allocation $\tilde{\mathbf{p}}$. Theorem 2 implies that, at the vNM utility profile $\hat{\mathbf{u}}$, the difference in *cardinal* utility between $\tilde{\mathbf{p}}$ and $\hat{\mathbf{p}}$ is bounded above by the difference of their *ordinal* efficiency deficits. Thus, we arrive at a general relationship between these two measures of ordinal and cardinal efficiency.

Moreover, when $\hat{\mathbf{p}}$ is ordinally efficient, we have that

$$\mathbf{u}(\hat{\mathbf{x}})'(\tilde{\mathbf{p}} - \hat{\mathbf{p}}) \leq D(\tilde{\mathbf{p}}) - D(\hat{\mathbf{p}}) = D(\tilde{\mathbf{p}}) \leq 0,$$

so that the allocation $\hat{\mathbf{p}}$ is immediately seen to maximize ex-ante welfare at $\hat{\mathbf{u}}$, which, we recall, is compatible to the underlying ordinal preferences. Thus, we arrive at a related, though slightly different, proof of the ordinal efficiency welfare theorem.

5 Directions for Future Research

The results in this paper provide a concise characterization of ordinal efficiency. In particular, an allocation is ordinally efficient if and only if its ordinal efficiency deficit (OED), a piecewise-linear convex function on the set of allocations, is zero. We believe that this insight, coupled with the more general optimization framework explored in this work, may prove useful in future research in random-assignment and house-allocation models. In particular, one may frame all sorts of existence questions by setting up a trivial optimization problem (i.e., one with a zero objective), imposing as constraints desired properties of efficiency, equity, and voluntary participation, and examining its dual. A similar approach may be helpful in the comparison of individual allocation mechanisms;

in particular, one can attempt to provide bounds on the difference of their ex-ante welfare, for a range of preference-compatible utility profiles.

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